THERMODYNAMIC PROPERTIES OF ICE, WATER, AND A MIXTURE OF THE TWO AT HIGH PRESSURES

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Water is a substance with a large variety of crystalline forms [1-4]. In natural conditions, hexagonal ice Ih, in which water crystallizes at a temperature of T = 273.15 K and a pressure of $p = 1.013 \cdot 10^5$ Pa, is the thermodynamically stable phase. In the propagation of a compression wave which arises under the effect of explosive and impact loads, as the experiments in [5, 6] show, phase transition processes take place in Ih ice. In particular, in a certain range of changes in the pressure applied to a sample of Ih ice, melting of the ice in the compression wave and the formation of a two-phase Ih ice—water mixture are observed. The theoretical description of the melting process in propagation of a compression wave over the ice within the framework of the mechanics of continuous media is based on the study of the thermodynamic properties of ice, water, and a mixture of the two [7, 8].

The equation of state of water has been studied in detail at a temperature above 273 K [9]. The anomalous properties of water at a temperature below 273 K and the possibility of phase transitions at high pressures into different modifications of ice create difficulties in establishing the equation of state of water in this temperature range. Nevertheless, recently conducted experiments by man investigators make it possible to reestablish the thermal equation of state (TES) of water, ice, and a mixture of them at negative temperatures based on indirect data.

1. Thermodynamic Properties of Ih Ice.

In studying the thermodynamic properties and equilibrium melting of ice, we will use the pressure p and temperature T as independent variables.

In the case of propagation of a compression wave in a solid isotropic deformable medium, in addition to the pressure, it is necessary to consider the stress tensor deviator. The nonhydrostatic character of the stress tensor and the difference, for example, in the longitudinal stress σ_1 in a plane wave from p are significant when the stresses which develop in the medium do not exceed the yield point of the substance τ_*. For Ih ice, the tensile strength and yield point in normal conditions do not exceed several MPa [10-12], while the effects of melting are markedly manifested at pressures of the order of 10-100 MPa [5, 6]. In this respect, we will neglect the shear stresses and consider p the only important component of the stress tensor (hydrostatic approximation) in the present study.

The TES of ice in the hydrostatic approximation is

$$V_1 = V_1(p, T) \tag{1.1}$$

(V_1 is the specific volume of the ice). Subscript 1 corresponds to values describing ice and subscript 2 indicates values describing water. Equation (1.1) can be constructed with the data from experimental measurements of the thermodynamic volume expansion coefficient $\alpha_{T1} = V_1^{-1}(\partial V_1/\partial T)_p$ and the isothermal compressibility coefficient $\beta_{T1} = -V_1^{-1}(\partial V_1/\partial p)_T$ by integration of the differential equation

$$dV_1/V_1 = -\beta_{T1}dp + \alpha_{T1}dT. \tag{1.2}$$

The selection of the method of integration is based on the analysis of the P—T diagram of ice plotted with the static measurements in [13, 14]. In Fig. 1, curves 1 and 2 are melting of Ih ice and III ice, and curve 3 is the phase equilibrium of Ih ice—III ice. Point P is the triple point of water at $p = 2.07 \cdot 10^2$ MPa and T = 251.15 K. We note that Ih ice, in contrast to other modifications of ice, is a material in which the melting point decreases with an increase in the pressure. For determining $V_1 = V_1(p, T)$, it is convenient

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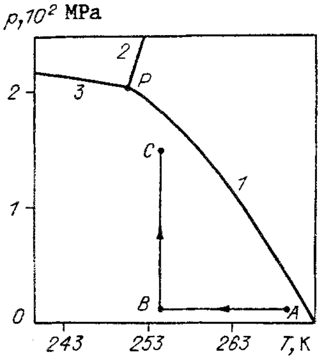

Fig. 1

TABLE 1

p, MPa	T, K	$V_1\cdot10^3$, m^3/kg	
		from [13]	from (1.9)
0	273,15	1,0900	1,0900
49,1	269,05	1,0775	1,0831
98,1	264,45	1,0684	1,0764
147,2	259,15	1,0616	1,0699
196,2	252,85	1,0571	1,0635

to select broken line ABC composed of segments AB and BC parallel to the coordinate axes, as the pathway of integration (Fig. 1). The point corresponding to $p_0 = 10^5$ Pa and $T_0 = 273.15$ K was selected as initial point A, then $V_{10} = 1.091$ m³/kg. As a result of integration of the complete differential (1.2), we obtain the dependence

$$V_1(p, T) = V_{10}\exp\left\{-\int\limits_{p_0}^{p}\beta_{T1}(p', T)\,dp' + \int\limits_{T_0}^{T}\alpha_{T1}(p_0, T')\,dT'\right\}. \tag{1.3}$$

The temperature in °C is designated by T_*, and the temperature in Kelvin is designated by T: $T_* = T + 273.15$.

Let us examine the experimental data required for determination of $V_1(p, T)$. The dependence of the linear expansion coefficient of ice on T in the 243-273 K range is given in [15]. With the data in [15], we find

$$10^6\cdot\alpha_{T1}(p_0, T) = 157.56 + 0{,}5556T_* + 2.655\cdot10^{-2}T_*^2 + 7.11\cdot10^{-4}T_*^3. \tag{1.4}$$

The specific volume of ice at pressure p_0 is

$$V_1^0(T) = V_{10}\exp\left\{\int\limits_{T_0}^{T}\alpha_{T1}(p_0, T')\,dT'\right\}. \tag{1.5}$$

For finding β_{T1}, we will use the results of measurements of the elastic constants of Ih ice. The temperature dependence of the isothermal modulus of dilatation K_T(Pa) for $p = p_0$ is [16]

$$K_T = 86.47\cdot10^8(1 - aT_*)/(1 - aT_m)$$

($a = 1.418\cdot10^{-3}$, $T_m = -16$). Hence

$$\beta_{T1}(p_0, T) = \beta_{T1}^0(T) = \frac{1}{K_T} = \frac{1.1827\cdot10^{-10}}{1 - aT_*}. \tag{1.6}$$

Bridgman found a different value of β_{T1}^0 equal to $3.3\cdot10^{-10}$ Pa^{-1}, and it significantly differs from Eq. (1.6) and the data of other investigators [4]. Equation (1.6) was used for the calculation below.

The dependence of β_{T1} on the pressure must still be found. The experiments in [17] show that for Ih ice, the following relation holds a low temperatures in a wide range of changes in p

$$K_T = K_{T0} + m_1(p - p_0) \tag{1.7}$$

with coefficient m_1, which is weakly dependent on T. It was shown in [18] that $m_1 = 4.4$ for the temperature region of interest to us. That is,

$$\beta_{T1} = \frac{\beta_{T1}^0(T)}{1 + m_1\beta_{T1}^0(T)(p - p_0)}. \qquad (1.8)$$

Using Eqs. (1.3)-(1.8) and recalling that pressures of $p \gg p_0$ will subsequently be examined, we arrive at the thermal equation of state of ice

$$V_1(p, T) = \frac{V_1^0(T)}{\left[1 + m_1\beta_{T1}^0(T)\,p\right]^{1/m_1}}. \qquad (1.9)$$

Pounder [19] previously proposed a TES for ice

$$V_1^{-1} = 916.8\left[1 + 0.94\cdot10^{-12}(p - 10^5)\right](1 - 1.53\cdot10^{-4}T_*), \qquad (1.10)$$

where p is measured in Pa. It is easy to see that if $\beta_{T1}{}^0 = \text{const}$ in Eq. (1.9) and we assume that $\alpha_{T_1} \sim 10^{-4}\ \text{K}^{-1}$ and $\beta_{T1}{}^0 \sim 10^{-10}\ \text{Pa}^{-1}$, then for $p \sim 10$ MPa, after linearization we obtain an equation of type (1.10) from Eq. (1.9). Thus, Eq. (1.10), well known in the literature on the physics of ice, generalizes Eq. (1.9) to the case of a wider range of changes in p and T with consideration of the more precise values of the thermodynamic characteristics of ice.

The results of Bridgman's experiments [13] and the calculation of V_1 with Eq. (1.9) with $V_{10} = 1.09\cdot10^{-3}\ \text{m}^3/\text{kg}$ are reported in Table 1. Note that in comparison to the data in [13], Eq. (1.9) gives slightly overestimated values of V_1, although the relative error does not exceed 0.9%.

The specific heat capacity of Ih ice is

$$c_{p1}(p, T) = c_{p1}(p_0, T) - T\int_{p_0}^{p} V_1\left(\frac{\partial\alpha_{T1}}{\partial T} + \alpha_{T1}^2\right)dp'. \qquad (1.11)$$

The value of $c_{p1}(p_0, T)$ was measured in a wide range of T. In particular, the following dependence [20] is valid for the values of T from the examined range of 243-273 K

$$c_{p1}(p_0, T) = 2.115\cdot10^3 + 7.79T_*\ \ \text{J/(kg}\cdot\text{deg)}. \qquad (1.12)$$

The expressions for α_{T1} and $\partial\alpha_{T1}/\partial T$ are determined from Eq. (1.9).

The thermodynamic characteristics of Ih ice obtained were used below in studying equilibrium melting of ice.

2. Thermodynamic Properties of Water. The thermodynamic properties of water, including the TES, have been studied in detail at T > 273 K in a wide range of changes in the pressure [9, 21-23]. For the purpose of the present study, it is necessary to establish the properties of water at $T \leq 273$ K and $p \leq 2.07\cdot10^2$ MPa.

At different pressures from 0.1 to $3\cdot10^2$ MPa, water allows significant supercooling. The region of the supercooled liquid state of H_2O covers a significant part of the region of existence of Ih ice, and the lowest temperature of supercooled water is attained for $p = 2\cdot10^2$ MPa and is equal to 181 K [1]. In addition, water at T < 277 K has a number of anomalous properties: the volume expansion coefficient of water $\alpha_{T2}(p_0, T)$ is negative, the specific heat capacity $c_{p2}(p_0, T)$, and isothermal compressibility $\beta_{T2}(p_0, T)$ increase with a decrease in the temperature. An explanation of these anomalies is found in the unique properties of the H_2O molecule [1].

Measurements of the density of supercooled water to T = 243 K were conducted in [24]. Analogous data up to T = 253 K are reported in [25]. Within the limits of the experimental precision, [24] and [25] are in agreement.

The results of the precision experiments to determine α_{T2} in a wide range of changes in the temperature and pressure are given in [26]. The interpolation equation which gives $\alpha_{T2}(p, T)$ and describes its anomalies is [26]

$$\begin{gathered} 10^4\alpha_{T2}(p,\ T) = A + B/(C + \Pi), \\ A = a_1 + a_2T + a_3T^2,\ B = a_4 + a_5T + a_6T^2 + a_7\Pi T + a_8\Pi, \\ C = a_9 + a_{10}T + a_{11}T^2 + a_1T^3,\ \Pi = p + a_{13}p^2 + a_{14}p^3 \end{gathered} \qquad (2.1)$$

[coefficients a_i and a_{ij} from Eq. (2.1) are reported in the Appendix].

The isothermal compressibility of water is determined with Eq. (2.1) as

$$\beta_{T2}(p, T) = \beta_{T2}(p, T_1) - \int_{T_1}^{T} \left(\frac{\partial \alpha_{T2}}{\partial p}\right)_T dT'. \tag{2.2}$$

The dependence of $\beta_{T_2}(p, T_1)$ for $T_1 = 323.16$ K was established in [26] by processing the data in [27]:

$$10^4 \beta_{T2}(p, T_1) = \sum_{i=0}^{4} b_i p^i. \tag{2.3}$$

Coefficients b_i are reported in the Appendix. The value of β_{T_2} in Eqs. (2.2) and (2.3) is measured in bar^{-1}, and the pressure p is measured in bars.

The TES of water is given by the equation

$$V_2(p, T) = V_{20} \exp\left\{-\int_{p_0}^{p} \beta_{T2}(p', T_1)\, dp' + \int_{T_1}^{T} \alpha_{T2}(p, T')\, dT'\right\} \tag{2.4}$$

($V_{20} = 1.01215 \cdot 10^{-3}$ m^3/kg). In the present study, the specific volume of water was determined numerically according to Eq. (2.4) with the Simpson equation. We note that Eq. (2.4) describes the anomalous properties of water noted above.

TES of water which is less precise but satisfactory from a practical point of view can be obtained with the results in [25]. Interpolation equations for $\beta_{T_2}(p_0, T)$ and $V_2{}^0 \times (T) = V_2(p_0, T)$ when T changes from 273 to 253 K are given in this study. If we assume that $\partial(\beta_{T_2}{}^{-1})/\partial p = m_2$ for $m_2 = 6.7$ [28], then similarly to Sec. 1, we arrive at the equation

$$V_2(p, T) = \frac{V_2^0(T)}{[1 + m_2 \beta_{T2}(p_0, T)\, p]^{1/m_2}}. \tag{2.5}$$

The numerical coefficients of Eq. (2.5) are given in the Appendix. The values of the specific volume calculated with Eq. (2.5) for T from 273 to 253 K and p up to $2 \cdot 10^2$ MPa were compared with Bridgman's experimental data on the specific volume of water in [13, 14]. The maximum relative error does not exceed 0.2% in the indicated range of changes in p and T, which demonstrates the practical validity of Eq. (2.5).

The specific heat capacity of water $c_{p2}(p, T)$ is established by the dependence

$$c_{p2}(p, T) = c_{p2}(p_0, T) - T \int_{p_0}^{p} V_2 \left(\frac{\partial \alpha_{T2}}{\partial T} + \alpha_{T2}^2\right) dp'. \tag{2.6}$$

The expressions for the functions in Eq. (2.6) under the integral sign are determined from Eqs. (2.1) and (2.4), and the integral itself is calculated with the Simpson equation. The first term in the right part of Eq. (2.6) can be found with the experimental data on supercooled water. It was noted in [29] that for $p = p_0$ the specific heat capacity of water c_{V2} is almost constant in the range of changes in T from 247 to 278 K and is equal to $4.206 \cdot 10^3$ J/(kg·deg). Then $c_{p2}(p_0, T)$ is found with the well-known thermodynamic equation

$$c_{p2} - c_{V2} = T \alpha_{T2}^2 V_2 / \beta_{T2}, \tag{2.7}$$

whose right part is calculated with Eqs. (2.1)-(2.4). Calculation of $c_{p2}(p_0, T)$ with Eq. (2.7) shows that this value increases with a decrease in T, which corresponds to the experiments from [29].

3. Calculation of Equilibrium Melting of Ih Ice. Line 1 in Fig. 1 corresponds to the p and T at which the coexistence of Ih ice and water in the form of a two-phase mixture is thermodynamically admissible. Let z be the weight fraction of water in the mixture. Then the specific volume of the mixture V and its entropy S are written as

$$V = (1 - z)V_1 + zV_2, \quad S = (1 - z)S_1 + zS_2 \tag{3.1}$$

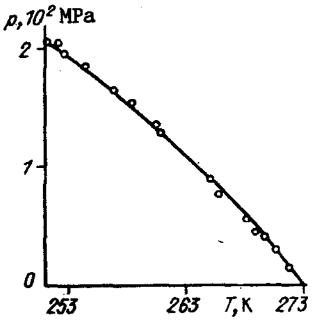

Fig. 2

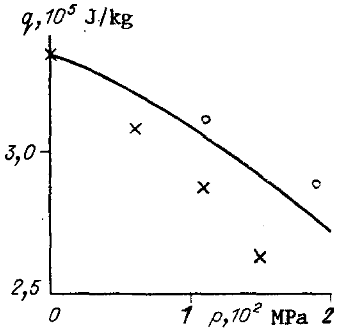

Fig. 3

(S_1 and S_2 are the entropy of ice and water). A correlation was established in [30] between the thermodynamic parameters of the phases and the mixture:

$$(\partial V/\partial p)_S = (1 - z)[(\partial V_1/\partial p)_T + (2T/q)(\partial V_1/\partial T)_p \times \tag{3.2}$$
$$\times (V_2 - V_1) - (c_{p1}T/q^2)(V_2 - V_1)^2] + z[(\partial V_2/\partial p)_T + (2T/q)(\partial V_2/\partial T)_p (V_2 - V_1) - (c_{p2}T/q^2)(V_2 - V_1)^2]$$

(q is the specific heat of melting of Ih ice). The adiabatic compressibility of the mixture (3.2) plays an important role in the study of compression wave propagation accompanied by a phase transition with respect to the question of the stability of the wave configurations formed. A different equation for $(\partial V/\partial p)_S$ which requires explicit expression of S_1 and S_2, is proposed in [31], where the properties of bismuth were studied.

Let us examine the process of adiabatic loading of the mixture, considering S = const. The partial derivative in the left part of Eq. (3.2) is then substituted by dV/dp. It is necessary to add another two equations to (3.2) for determining the dependence of V on p on the phase equilibrium line: the Clapeyron–Clausius equation, which we will write as

$$dT/dp = T(V_2 - V_1)/q, \tag{3.3}$$

and the equation which describes the change in the specific heat of melting along the phase equilibrium curve [32]:

$$\frac{dq}{dp} = \left[c_{p2} - c_{p1} + \frac{q}{T} - \frac{q\,(V_2\alpha_{T2} - V_1\alpha_{T1})}{V_2 - V_1}\right]\frac{T\,(V_2 - V_1)}{q}. \tag{3.4}$$

We will thus have a system of ordinary differential Eqs. (3.2)-(3.4); after solving it together with Eq. (3.1) for the given initial conditions, we can determine the adiabatic dependence V(p), and numerically calculate T(p) (i.e., determine the melting curve) and find q(p). All of the values in the right parts of system (3.2)-(3.4) were obtained in Secs. 1, 2 of the present study.

System (3.1)-(3.4) was integrated numerically with the Runge–Kutta method. The starting data for $p = p_0$ were $T = 273.15$ K and $q = 333.7 \cdot 10^3$ J/kg. The initial volume of the mixture V varied from $V_0 = 1.09075 \cdot 10^{-3}$ m³/kg, which corresponds to ice for $z = 0$, up to $V_0 = 1.0182 \cdot 10^{-3}$ m³/kg (mixture of ice and water for $z = 0.8$). The phase boundaries of the mixture in plane p–V was determined simultaneously with the calculation of the dependences V(p), T(p), and q(p) during integration: left $V = V_1(p, T(p))$ and right $V = V_2(p, T(p))$. The results of the calculations are shown in Figs. 2-4.

The calculated melting curve T = T(p) is illustrated by the solid line and Bridgman's experimental data are represented by the circles in Fig. 2 [13]. The calculated and experimental data are in good agreement. An equation usually used for the analytical description of the melting curve is proposed in [28]:

$$p = -3{,}952 \cdot 10^2[(T/273.16)^9 - 1],\ [p] = \text{MPa}. \tag{3.5}$$

The difference in the values of T(p) calculated with Eq. (3.5) from the results shown in Fig. 2 does not exceed 0.5 K in the entire pressure range examined.

The dependence of the specific heat of melting q on p is illustrated by the solid line in Fig. 3. The data from [13] are indicated by the crosses, and the data from [33] are indicated by the circles. The q calculated in the present study have intermediate values and

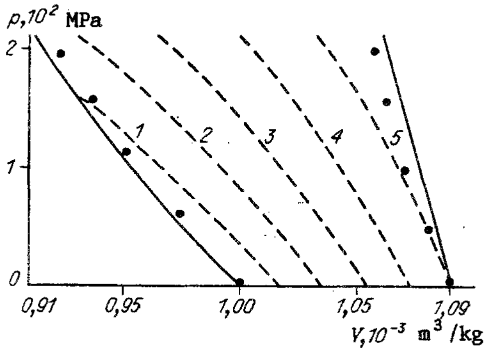

Fig. 4

differ from Bridgman's q by 14% for $p = 2.07 \cdot 10^2$ MPa. The difference in the values of V_1 on the melting curve found here and reported in [13] (see Table 1) is perhaps the cause of this difference.

The results of calculating V as a function of the initial conditions are shown in Fig. 4. The right solid line is the right boundary of the region corresponding to the Ih ice–water equilibrium mixture, and the left solid line corresponds to the left boundary of the two-phase region. The isentropes of the mixture for different initial values of the fraction of water z in the mixture for $p = p_0$ and $T = 273.15$ K are applied on the broken lines: 1-5) $z = 0.8, 0.6, 0.4, 0.2, 0$. The experimental data in [13] for the specific volume of water and Ih ice on the melting curve are indicated by the points. The final value of the fraction of water z on curves 2-5 is 0.88, 0.65, 0.49, and 0.19, respectively. If pure ice was initially at atomospheric pressure and $T = 273$ K, then approximately 20% of the mass of the ice changes into water with an adiabatic increase in the pressure to $2 \cdot 10^2$ MPa. If the initial mixture contained more than 80% water, then it will all change into water at some pressure as a result of loading.

The calculated isentropes of the mixture, as Fig. 4 shows, have the property $(\partial^2 V/\partial p^2)_S < 0$. An analogous situation (see [29]) occurs in the case of equilibrium melting of bismuth 1, resulting in instability of the shock wave (SW) which propagates in the bismuth, and its transformation into a continuous compression wave.

A thermal equation of state of Ih ice at a negative temperature in a wide range of pressure changes was thus constructed in the present study based on an experiment on the volume expansion and isothermal compressibility coefficients. This equation is in agreement with the Pounder equation of state for low pressures and with Bridgman's experimental data. The specific heat capacity of ice c_{p1} was determined. The results of the precision experiments on isothermal compressibility in [26] permitted deriving the thermal equation of state of water at $T < 273$ K. The specific heat capacity of water whose temperature dependence corresponds to the experiment was calculated. The features established extend the existing concepts on the properties of water [9, 21-23] to the range of a change in the temperature of $T < 273$ K.

The equilibrium melting of Ih ice was investigated. The adiabatic dependence of the specific volume on the pressure was found for the ice–water mixture. The melting curve was numerically calculated. The phase boundaries of the Ih ice–water mixture were determined in the p–V diagram. The results obtained are in agreement with Bridgman's experimental data. It was found that in adiabatic compression along the melting curve of pure ice in the initial state in normal conditions, approximately 20 wt. % of the ice turns into water below a pressure of $2 \cdot 10^2$ MPa. The calculated isentropes of the Ih ice–water mixture have the property of $(\partial^2 V/\partial p^2)_S < 0$, which should result in instability of propagation of SW in such a medium (the SW degenerates into a continuous compression wave).

The results of the study can serve as the basis for investigating propagation of compression waves and interpreting the experimental data on shock loading of ice.

Appendix. 1. The numerical coefficients in Eq. (2.1) are: $a_1 = 4.78506 \cdot 10^1$, $a_2 = -8.12847 \cdot 10^{-2}$, $a_3 = 8.49849 \cdot 10^{-5}$, $a_4 = 5.56047 \cdot 10^5$, $a_5 = -3.76355 \cdot 10^3$, $a_6 = 5.56395$, $a_7 = 5.59682 \cdot 10^{-3}$, $a_8 = -2.76522 \cdot 10^1$, $a_9 = -4.28076 \cdot 10^3$, $a_{10} = -3.39150 \cdot 10^1$, $a_{11} = 3.65873 \cdot 10^{-1}$, $a_{12} = -5.89617 \cdot 10^{-4}$, $a_{13} = 3.28892 \cdot 10^{-4}$, $a_{14} = -2.65933 \cdot 10^{-8}$.

2. The numerical coefficients in Eq. (2.3) are: $b_0 = 4.41753 \cdot 10^{-1}$, $b_1 = -1.09205 \times 10^{-4}$, $b_2 = 1.99785 \cdot 10^{-8}$, $b_3 = -2.08128 \cdot 10^{-12}$, $b_4 = 8.86050 \cdot 10^{-17}$.

3. The functional dependence and numerical coefficients in Eq. (2.5) are:

$$10^3 \cdot V_2^0(T) = (1 + b_1 T_*) \Big/ \sum_{n=0}^{5} a_n T_*^n, \quad 10^{11} \cdot \beta_{T2}(p_0, T) = \sum_{n=0}^{5} c_n T_*^n.$$

$V_2{}^0(T)$ is measured in m^3/kg, and β_{T2} is measured in Pa^{-1}; $b_1 = 18.159725 \cdot 10^{-3}$; $a_0 = 0.9998396$, $a_1 = 18.224944 \cdot 10^{-3}$, $a_2 = -7.92221 \cdot 10^{-6}$, $a_3 = -55.44846 \cdot 10^{-9}$, $a_4 = 149.7562 \times 10^{-12}$, $a_5 = -393.2952 \cdot 10^{-15}$, $c_0 = 50.9804$, $c_1 = -0.374957$, $c_2 = 7.21324 \cdot 10^{-3}$, $c_3 = -64.1785 \cdot 10^{-6}$, $c_4 = 0.34302 \cdot 10^{-6}$, $c_5 = -0.684212 \cdot 10^{-9}$.

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SPECTRAL MODELS FOR CASCADE PROCESSES IN HOMOGENEOUS TURBULENCE

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There is a long history [1, 2] of simulating energy transport in homogeneous isotropic turbulence by means of spectral-density equations. Each model to some extent reflects the energy transport over the spectrum, but those spectral models usually do not reflect the cascade transport, i.e., sequential transport via nearest neighbors in the spectrum. That specific feature of turbulent transport is closely reflected by a reduction model due to Obukhov, Desnyanskii and Novikov, Gledzer, and so on [3-5]. It is of interest to demonstrate cascade transport directly on a spectral model. Here we propose such a model for homogeneous isotropic turbulence.

The stationary Kolmogorov—Obukhov state (−5/3 law) is obtained in a scale-invariant range, together with the nonstationary state having spectral density $E(k, t) \sim t^{-2}k^{-3}$. In the latter, there is energy transfer from the small-scale pulsations to the large-scale ones, which is usually ascribed to two-dimensional turbulence [6]. That state is observed also in lattice turbulence, which was used to simulate two-dimensional in the [7] experiments. In the dissipative range (in the short-wave limit), the model leads to a spectrum $E(k) \sim \exp - ak$, which with logarithmic accuracy coincides with the Kreichnan—Kuz'min—Patashinskii asymptote [8].

I have calculated the damping for the total pulsation energy and the increase in the integral scale for the initial conditions $E_0(k) \sim k^m \exp -(k/k_0)^2$. The result is $\overline{u}^2 \sim t^{-n}$, $L \sim t^p$, in which $n = 2(1 + m)/(3 + m)$; $p = 2/(3 + m)$; and L is the integral turbulence scale. For m = 1-4 corespondingly, those formulas give n = 1, p = 1/2; n = 1.2, p = 0.4; n = 4/3, p = 1/3; n = 10/7, p = 2/7, i.e., values familiar from experiments and various theories [1, 9-14].

Hypotheses on the vortex interaction in turbulent flows are frequently formulated as spectral transport functions T(E; k, t) in the equation for such transport [1]

$$\partial E(k, t)/\partial t = -2\nu k^2 E(k, t) + T(E; k, t) \tag{1}$$

in which T(E; k, t) is a function of k and t and a functional of E(k, t). This incorporates inertial energy transport. The explicit form of that function-functional is unknown. We expand T(E; k, t) as a functional series in powers of E(k, t), and as the inertial effects are nonlinear, the series will be analogous not to a Taylor series but instead to an expansion near a branch point:

$$T(E;\ k, t) = \sum_{n=0}^{\infty} \int_0^{\infty} dk_1 \ldots dk_n G(k;\ k_1, \ldots, k_n)\, E^{1/m}(k_1, t) \ldots E^{1/m}(k_n, t). \tag{2}$$

Here m is a positive number (the algebraic order of the branch point), while $G(k;\ k_1, \ldots, k_n)$ describes the inertial effects from vortices having scales $k_1^{-1}, \ldots, k_n^{-1}$ on vortices having scales k^{-1}.

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